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Relaxation-Time Approximation for a Boltzmann Gas in Robertson–Walker Universe Models

Zbigniew Banach[1] and Hanna Makaruk[2]

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Given Einstein's theory of gravitation, the relaxation-time approximation for a general-relativistic Boltzmann equation is studied with a view to demonstrating its usefulness in the context of Robertson–Walker universe models. Solutions of the full nonlinear equations for the metric and the distribution function are examined, together with their relation to linearized perturbations. Emphasis is placed on finding analogs of the exact results on strong or weak convergence to equilibrium employed in special-relativistic kinetic theory. At the late stages of cosmic expansion, an explicit choice of the empirical collision frequency is made to fit optimally the relaxation-time model to the "actual" Einstein–Boltzmann system. Finally, perspectives for some future generalizations are outlined.

1. INTRODUCTION

At the early and late stages of cosmic expansion, the Einstein–Boltzmann coupled system of equations can be used to describe the time evolution of a collision-dominated one-component gas of massive particles. Solving this system is not simple, but it turns out that there exists a straightforward approximate scheme when the matter is only slightly perturbed away from the background Robertson–Walker cosmological model. Indeed, under these circumstances, the linear perturbation method is a very effective mathematical technique which allows a study of general-relativistic dynamics to be feasible. With such a method, the metric and the distribution function may be determined directly from an analysis of the linearized Einstein–Boltzmann system (see, e.g., Banach and Piekarski, 1994a–c). Accordingly, it seems also

[1] Centre of Mechanics, Institute of Fundamental Technological Research, Department of Fluid Mechanics, Polish Academy of Sciences, Swietokrzyska 21, 00-049 Warsaw, Poland.
[2] Centre of Physics and Applied Mathematics, Institute of Fundamental Technological Research, Polish Academy of Sciences, Swietokrzyska 21, 00-049 Warsaw, Poland.

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important to consider the validity of perturbation techniques by finding the properties of solutions of the full nonlinear Einstein–Boltzmann system near the Robertson–Walker solutions, together with their relation to solutions of the linearized field equations.

Unfortunately, as already remarked, the original Einstein–Boltzmann equations are not easy to solve and to analyze, and elementary inspection shows that the difficulty in solving the Boltzmann equation is largely due to the complicated structure of the nonlinear collision term. Consequently, in order to examine the question concerning the range of validity of perturbation theories, it is very tempting to try guessing model equations with the same basic features as the Boltzmann equation, but simpler to solve. For cosmological problems, a considerable but nontrivial simplification can be achieved if one postulates that the collision term takes the Bhatnagar–Gross–Krook form (Bhatnagar *et al.*, 1954). In this case, the time evolution of the metric and of the distribution function is governed by the general-relativistic system of equations, which we call the Einstein–Krook system.

One major reason for investigating the properties of the Einstein–Krook system is as follows: the resulting nonlinear theory may be compared with perturbation theory in a much closer way than the original Einstein–Boltzmann theory. To illustrate this, we shall give an example. The simplest situation is when the space-time geometry is that of a $k = 0$ Robertson–Walker spacetime. Then the discussion of the linearized Einstein–Boltzmann system in the late universe allows one to find a necessary and sufficient condition under which every small perturbation of the equilibrium distribution function, either classical or relativistic, decays to zero as time goes on (Banach and Piekarski, 1994a). However, one cannot easily obtain a corresponding condition for the full nonlinear Einstein–Boltzmann system, because such a system is far less amenable to analytic solution than is its linearized version. On the other hand, a systematic description of the time evolution of a Boltzmann gas in terms of model equations may give the distinct possibility of gaining qualitative insight into the more complicated nonlinear regime.

In this paper, the Einstein–Krook system is studied with a view to a deeper understanding of the connection between linearized solutions and solutions of the full nonlinear equations. For simplicity, we assume that the metric is of Robertson–Walker form. It is shown that if certain plausible inequalities are satisfied, then the Einstein–Krook system drives the molecular density for a one-component gas of massive particles toward a Maxwellian distribution of microscopic momenta. Moreover, we consider the linearized equations to infer that physically reasonable solutions of these equations do indeed correspond to solutions of the full nonlinear system. Finally, we compare both of these models with one based on the Einstein–Boltzmann theory which has already proved useful in a cosmological setting (Banach and Piekarski, 1994c).

At first sight, the relaxation-time approximation for a Boltzmann gas is an *ad hoc* postulate which does not arise from a rigorous starting point. Nevertheless, under conditions appropriate to the introduction of perturbation theory, one reasonable method of obtaining a molecular formula for the empirical collision frequency would be to guess its form using such guidance as is available from a discussion of the linearized Boltzmann equation. The agreement with complete theory is found to be very good in this case, and one can see from the results that the linearized Einstein–Krook system exhibits the same basic features as the corresponding Einstein–Boltzmann system if the empirical collision frequency is adequately chosen. This in turn shows that the problem of the trend to equilibrium for Robertson–Walker universe models is not an artifact of using a phenomenological theory such as the relaxation-time kinetic theory.

Another remark is also in order. Even if the metric were not exactly isotropic and spatially homogeneous, as would be the case for perturbations near the Robertson–Walker data, the knowledge of properties of the Einstein–Krook model may still offer specific advantages in the exploration of some of the different aspects of physical cosmology. Among the problems that can be studied with this sort of approach, the examination of the effect of inhomogeneities on the time development of perturbations presents a most interesting challenge. Thus, for example, it would be important to provide a rigorous derivation of the Jeans instability via a linearized Einstein–Krook system and to show explicitly that the evolution of the distribution function is affected by particle horizons. Clearly, we here concern ourselves with situations where the pressure does not vanish in the background; otherwise the Jeans criterion is irrelevant to this evolution, whether we consider large- or small-scale inhomogeneity, because the individual world lines evolve independently (see, e.g., Banach and Piekarski, 1994c, Section 6). In any case, the relaxation-time approximation will find its most interesting applications in considerations of the distribution function for large values of the wave vector, away from the usual hydrodynamic regime where more conventional methods (relating collision times to representative macroscopic times) are successful.

We here proceed as follows. To prepare for the discussion, Section 2 specializes the Einstein–Krook system to the case of homogeneous and isotropic model universes, flat and positively or negatively curved. Given such a specialization, Section 3 in turn formulates a simple theorem of the weak trend to equilibrium. Section 4 first derives the linearized perturbation equations and then compares the predictions of the Krook and Boltzmann theories in a physically interesting context. Section 5 indicates the direction of future research.

One final word regarding this paper. Some of the results reported in Section 3, especially those based on strong or weak convergence to equilibrium, were completed before April 1993.

2. DESCRIPTION OF THE EINSTEIN–KROOK SYSTEM

In the Robertson–Walker spaces, one can choose coordinates so that the metric has the form (Misner *et al.*, 1973)

$$ds^2 = -c^2\,dt^2 + [R(t)/W]^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{2.1a}$$

where

$$W := 1 + (k/4)[(x^1)^2 + (x^2)^2 + (x^3)^2] \tag{2.1b}$$

and where k is the (constant) spatial curvature. By an appropriate choice of units, the value of k can be made to be $+1$, -1, or 0. The corresponding solutions for the expansion factor $R(t)$ represent, respectively, spaces of positive or negative curvature or flat space.

We shall consider only a simple gas, i.e., an assemblage of material particles with a continuous distribution function $f(t, p^r)$, all having the same proper mass m. The components of the particle four-momentum with respect to a local orthonormal tetrad $\{c^{-1}\,\partial/\partial t,\ (W/R)\,\partial/\partial x^r\}$ will be denoted by p^α. In our notation, Greek indices range from 0 to 3, Latin indices from 1 to 3. We assume for concreteness that $f(t, p^r)$ depends on p^r through $q := (p^r p_r)^{1/2}$. Then the Boltzmann equation may be written as

$$\frac{\partial f}{\partial t} - Hq\,\frac{\partial f}{\partial q} = J(f) \tag{2.2}$$

Here $J(f)$ is a collision term and H is Hubble's parameter defined by $H := \dot{R}/R$.

Since Boltzmann's expression for $J(f)$ is quite formidable, a more phenomenological approach has often been preferred (Bhatnagar *et al.*, 1954), based on the use of a relaxation-time approximation of the collision term,

$$J(f) = -\upsilon(f - \mathbb{F}) \tag{2.3}$$

where all the physics is supposed to be included in a single function $\upsilon(t)$ that should be evaluated elsewhere with a specific model. We name υ the empirical collision frequency. In relativistic theory, one characterizes $\mathbb{F}$ by

$$\mathbb{F} := (2\pi\hbar)^{-3}\exp[(\mu - cp^0)/k_B\vartheta] \tag{2.4}$$

This is, of course, the Maxwell–Jüttner molecular density (Synge, 1957). The question then arises of how to fit the parameters $\mu(t)$ and $\vartheta(t)$ to the actual $f(t, q)$. For our nonequilibrium gas, μ and ϑ are constrained to satisfy the conditions

$$\int_0^\infty q^2 \mathrm{F}\, dq = n/4\pi, \qquad \int_0^\infty p^0 q^2 \mathrm{F}\, dq = e/4\pi c \tag{2.5}$$

in which n is the number density and e is the energy per unit volume:

$$n := 4\pi \int_0^\infty q^2 f\, dq, \qquad e := 4\pi c \int_0^\infty p^0 q^2 f\, dq \tag{2.6}$$

Interpreting $\vartheta(t)$, we call this object the relativistic temperature of a gas at the time t. Clearly, the fitting of F to f is nonlinear in f. Indeed, when we come to apply (2.5) to (2.6), we recognize that μ and ϑ are complicated functionals of f. Nevertheless, only under the foregoing conditions does the resulting nonlinear model exhibit all the elementary, qualitative features of the Boltzmann equation; specifically, it admits an easily proved H-theorem.

The Einstein equations for the expansion factor $R(t)$ given in (2.1a), with the energy density of equations (2.6) and the pressure defined by

$$p := (4\pi c/3) \int_0^\infty (1/p^0) q^4 f\, dq \tag{2.7}$$

may be written as

$$H^2 = (8\pi G/3c^2)e - kc^2R^{-2} \tag{2.8a}$$

$$\dot{H} + H^2 = -(4\pi G/3c^2)(e + 3p) \tag{2.8b}$$

where G is Newton's gravitational constant. The equation of conservation of energy takes the form

$$\dot{e} + 3(e + p)H = 0 \tag{2.9}$$

This equation, of course, follows directly from (2.8) and is also consistent with the properties of a relaxation-time approximation of the collision term. As to the conservation equation for the number density n, we easily find from (2.2) and (2.5) that

$$nR^3 = \text{const} \tag{2.10}$$

With all these definitions in mind, the coupled system of governing equations for f and R consists of (2.2) and (2.8a), and the remaining cosmological equations are automatically fulfilled because of (2.5). Here and henceforth, we shall refer to (2.2) and (2.8a) as the Einstein–Krook system.

3. ELEMENTARY PROPERTIES OF THE RELAXATION-TIME MODEL

3.1. Some Useful Transformations

In order to integrate completely a simple relaxation-time kinetic equation

$$\frac{\partial f}{\partial t} - Hq\frac{\partial f}{\partial q} = -\upsilon(f - \mathbb{F}) \tag{3.1}$$

it is necessary to be able to determine the dynamical evolution of four variables: $R(t)$, $\mu(t)$, $\vartheta(t)$, and $\upsilon(t)$. In principle, after specifying υ, this evolution is obtainable from a rigorous analysis of the Einstein–Krook system. However, since the parameters μ and ϑ depend intricately on f, we must reject the possibility that we could solve equations (2.8a) and (3.1) analytically for R and f; and even if we could, the results would be difficult to describe and to interpret. Fortunately, at the late stages of cosmic expansion, there are certain relations which it is physically reasonable to assume for the gas-state variables. These will be discussed in this section. It turns out that in many circumstances these are sufficient to prove the existence of the trend to equilibrium, independent of the exact form of R, μ, ϑ, and υ.

First, if the approximate condition

$$\vartheta R^2 = \text{const} \tag{3.2}$$

illustrates the dependence of ϑ on R in the late universe (Peebles, 1980), then it is convenient to define the "nonrelativistic" temperature T as follows:

$$T := bk_{\mathrm{B}}^{-1}R^{-2} \tag{3.3}$$

In this equation, b is a positive constant which can be calculated uniquely by requiring that the quantity

$$\Gamma := T/\vartheta \tag{3.4}$$

evolves toward one. Clearly, Γ measures the relative size of the nonrelativistic temperature compared to the relativistic temperature. With such a definition of T, the result (2.10) changes into an alternative form

$$n(2\pi mk_{\mathrm{B}}T)^{-3/2} = \text{const} \tag{3.5}$$

Now implement a time-dependent canonical transformation from the original variable q to a new variable z obeying

$$z := (2mk_{\mathrm{B}}T)^{-1/2}q \tag{3.6}$$

Turning our attention back to (3.1), we then see that $f(t, z)$ will satisfy a transformed equation of the form

$$\frac{\partial f(t, z)}{\partial t} = -\nu(t)[f(t, z) - \mathbb{F}(t, z)] \tag{3.7}$$

where, as before, $\mathbb{F}$ stands for the Maxwell–Jüttner distribution function.

The next stage in the discussion is to express $\mathbb{F}$ in terms of z and t. After a fair amount of algebraic calculation, equations (2.5) may be rewritten as

$$\exp(\mu k_B \vartheta) = (2\pi\hbar)^3[4\pi m^2 c k_B \vartheta K_2(M)]^{-1} n \tag{3.8a}$$

$$e/mnc^2 = (3/M) + K_1(M)/K_2(M) \tag{3.8b}$$

where $K_r(M)$, $r = 1, 2, \ldots, \infty$, are the modified Bessel functions of the second kind; M is linearly proportional to the inverse of ϑ:

$$M := mc^2/k_B\vartheta \tag{3.9}$$

Further, introduce the dimensionless factor

$$\kappa := k_B T/mc^2 \tag{3.10}$$

Using the obvious identity $p^0 = mc(1 + 2\kappa z^2)^{1/2}$, we then find from (2.4) and (3.8a) that

$$\mathbb{F} = \Omega \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left[-\frac{2\Gamma z^2}{1 + (1 + 2\kappa z^2)^{1/2}}\right] \tag{3.11}$$

Here Ω is given by

$$\Omega := \Gamma^{3/2}[\Theta_2(M)]^{-1} \tag{3.12}$$

where

$$\Theta_2(M) := (2M/\pi)^{1/2} \exp(M)\, K_2(M) \tag{3.13}$$

Equation (3.11) has the advantage that, for $\Omega = \Gamma = 1$ and $\kappa = 0$, the Maxwell–Jüttner distribution function $\mathbb{F}$ takes the very simple form

$$\mathbb{F} = F := \frac{n}{(2\pi m k_B T)^{3/2}} \exp(-z^2) \tag{3.14}$$

Such a form is called a Maxwellian molecular density.

We are now ready to examine the consequences of using the above formulas in the context of (3.1). This could hardly be achieved with the full nonlinear Boltzmann equation, even though the present type of analysis has been successfully applied to the linearized Einstein–Boltzmann system (see, e.g., Banach and Piekarski, 1994c).

3.2. Evolution of the Distribution Function at Late Times

One possible way to determine the evolution of f is simply to integrate equation (3.7) from the time t_0 to the time t. It follows that, if $f_0 - \mathbb{F}_0$ is the initial value of $f - \mathbb{F}$, and if, moreover, the function $\mathbb{F}(t, z)$ is continuously differentiable with respect to t, it will always be possible to obtain a formula of the form

$$f(t, z) = \mathbb{F}(t, z) + \Delta(t_0, t)[f_0(z) - \mathbb{F}_0(z)] - \int_{t_0}^{t} \Delta(\sigma, t)\dot{\mathbb{F}}(\sigma, z)\, d\sigma \tag{3.15}$$

where an overdot indicates differentiation with respect to σ or t and $\Delta(\sigma, t)$ is defined by

$$\Delta(\sigma, t) := \exp\left[-\int_{\sigma}^{t} \upsilon(s)\, ds\right] \tag{3.16}$$

with σ ranging from t_0 to t.

Matter is treated here as an assemblage of material particles, which in the case of a chemically inert fluid might be a hydrogen gas during the dust-dominated epoch, for a redshift $Z \cong 1000$ until $Z \cong 30$ or so. Under these assumptions, we can compare the relaxation-time model with a full nonlinear Boltzmann equation. But the latter provides a most natural route to calculating the empirical collision frequency (discussed in Section 3.3 below) and to verifying that the rate of growth of υ^{-1} is no greater than that of the inverse of Hubble's parameter H. Thus the following condition arises which we call the kinetic-theory condition: There exist times t_0 and t_1, $t_1 > t_0$ such that

$$H(t) > 0, \qquad \upsilon(t) \geq H(t) \tag{3.17}$$

when $t \in [t_0, t_1)$. Hence, by combining (3.16) and (3.17), one obtains the largest upper bound for $\Delta(\sigma, t)$

$$\Delta(\sigma, t) \leq R(\sigma)/R(t) \tag{3.18}$$

where $t_0 \leq \sigma \leq t < t_1$.

What one must do now is to differentiate the function $\mathbb{F}$ as given by (3.11) with respect to time. After a bit of mathematical manipulation which employs only the following well-known inequalities [see also Synge (1957), equations (370) and (372), pp. 88, 89]:

$$1 + (15/8)M^{-1} \leq \Theta_2(M) \leq 1 + (15/8)M^{-1} + (105/128)M^{-2} \tag{3.19a}$$

and

$$-(15/8)M^{-2}[1 + (7/8)M^{-1}] \leq d\Theta_2(M)/dM \leq -(15/8)M^{-2} \tag{3.19b}$$

one finds from $\dot{\kappa} = -2\kappa H$ and $M = \kappa^{-1}\Gamma$ that

$$|\dot{\mathrm{F}}(t, z)| \leq 8\mathrm{F}(t, z)\left\{\left[\frac{1}{\Gamma(t)} + z^2\right]|\dot{\Gamma}(t)| + \left[\frac{1}{\Gamma(t)} + \Gamma(t)z^4\right]\kappa(t)H(t)\right\} \tag{3.20}$$

if $H > 0$ and $M > 1$.

Motivated by the above considerations, we can now formulate our main theorem.

Theorem. Let $t \in [t_0, t_1)$. Consider a situation in which $H > 0$, $\upsilon \geq H$, and $M > 1$. Assume further that there exist positive constants $\mathbb{B}$ and $\mathbb{C}$ such that

$$|\dot{\Gamma}(t)| \leq \mathbb{B}\kappa(t)H(t) = -\frac{1}{2}\mathbb{B}\dot{\kappa}(t) \tag{3.21a}$$

and

$$\Gamma(t) \leq \mathbb{C}, \qquad 1/\Gamma(t) \leq \mathbb{C} \tag{3.21b}$$

Then

$$|f(t, z) - \mathrm{F}(t, z)| \leq \mathcal{G}(t_0, z)[R(t_0)/R(t)] \tag{3.22a}$$

where [see equation (3.5)]

$$\mathcal{G}(t_0, z) := |f_0(z) - \mathrm{F}_0(z)| + 8n\mathbb{C}^{3/2}(2\pi m k_B T)^{-3/2} \times \exp\left\{-\frac{2\mathbb{C}^{-1}z^2}{1 + [1 + 2\kappa(t_0)z^2]^{1/2}}\right\}(1 + \mathbb{B})(\mathbb{C} + z^2 + \mathbb{C}z^4)\kappa(t_0) \tag{3.22b}$$

Proof. By (3.3), (3.10), and (3.18) we find that

$$\int_{t_0}^{t} \Delta(\sigma, t)\kappa(\sigma)H(\sigma)\, d\sigma \leq \kappa(t_0)[R(t_0)/R(t)] \tag{3.23}$$

Since $|\dot{\mathrm{F}}(t, z)|$ satisfies the inequality (3.20), the theorem follows from (3.11), (3.12), (3.15), (3.18), (3.19a), (3.21), and (3.23). ■

Remark. Any attempt to prove the validity of (3.21) is equivalent to solving the Einstein–Krook system; this would involve the complete analysis of (2.8a) and (3.7), and this we have already rejected. However, perturbation theory, which makes it possible in particular cases, will be explained in Section 4.

Turning to the inequality (3.22a), we now see that there is a trend in time: the object $|f(t, z) - \mathrm{F}(t, z)|$ can be bounded by a function which decreases with increasing t. Specifically, if the kinetic-theory condition holds for all times ($k = -1, 0$; $t_1 = \infty$; $\lim_{t\Rightarrow\infty} R(t) = \infty$), and if Ω and Γ approach asymptotically 1 (see Section 4), then $f(t, z)$ evolves toward a Maxwellian molecular density $F(z)$ as given by (3.14).

One might perhaps conjecture that this kind of behavior is somehow a particular feature of the relaxation-time approximation, especially in that the expression on the right-hand side of (3.7) contains υ and F. This, however, does not seem to be so. Other, more realistic kinetic equations are known to exhibit the same qualitative behavior (Banach and Piekarski, 1994a), and the possibility of obtaining a simple bound for $|f - \mathrm{F}|$ may be quite common to the Robertson–Walker universe models satisfying (3.17).

3.3. Identification of the Empirical Collision Frequency

Our kinetic-theory condition can be explicitly investigated only when the collision mechanism, as specified by the scattering cross section, is known. In the context of Boltzmann's equation, there will always be a preferred family of scattering cross sections representing the so-called relativistic hard interactions (Dudyński and Ekiel-Jeżewska, 1988). At the late stages of cosmic expansion, they give a relationship for υ in terms of the number density and the nonrelativistic temperature [see, e.g., equations (3.8b) and (5.9) in Banach and Piekarski (1994a)]. Precisely speaking, the best expression for υ, in the sense that it comes nearest to the Boltzmann equation, is given by

$$\upsilon = \lambda\sigma_0 n(4\pi^{3/2}m)^{-1}(2mk_B T)^{(1-j)/2} \tag{3.24}$$

where λ, σ_0, and j are constants ($\lambda > 0$, $\sigma_0 > 0$, $j \geq 0$). The value $j = 1$ arises, for instance, in the case of Maxwellian particles, repelling according to the inverse fifth power of the distance. For the hard-sphere model, in turn, we set [see footnote 6 in Banach and Piekarski (1994a)]

$$j = 0, \qquad \lambda \geq 0.7339(8\pi^2), \qquad \sigma_0 = 2r^2 \tag{3.25}$$

and characterize the quantity r by saying that $2r$ is the diameter of the particle.

The most important physical application of (3.24) and (3.25) is to atomic hydrogen at redshifts $Z \leq 1000$. After the epoch of the decoupling of matter and radiation, one would expect that the motion of massive particles can be considered to be collision-free. *But this is not always the case.* As an illustration, given (3.24) and (3.25), we have carefully verified that our kinetic-theory condition, namely the inequality $\upsilon \geq H$, can be used to study the time evolution of a hydrogen gas during the cosmic expansion from $Z \cong 1000$ to $Z \cong 30$.

We arrived at the condition $\upsilon \geq H$ from the attempts to formulate a theorem of the trend to equilibrium. However, problems are not so simple when the metric is not of Robertson–Walker form. In fact, one knows full well that density fluctuations in the early universe would develop into the irregularities we observe, and that there is no trend to equilibrium. Because of this, attention has been concentrated on the inhomogeneous *hydrodynamic* models, their relative merits, and possible tests. Nevertheless, our comments concerning the validity of $\upsilon \geq H$ in particular situations are not substantially modified by the presence of large- or small-scale inhomogeneity (or "dark matter" content of the universe) and are necessary to solve a methodological problem of physical cosmology illustrated by the following paradox: How are we to give a precise meaning to the statement that matter often behaves in an essentially hydrodynamic way, whereas the distribution function involves infinitely many degrees of freedom?

First of all, the temporal behavior of f is related to the value of this distribution at the initial time. But if the condition (3.17) is assumed to be correct over a large range of cosmic times, we can apply the inequality $\Delta(t_0, t) \leq R(t_0)/R(t)$ to (3.15) and so conclude that since Δ approaches zero, f is effectively expressible in terms of hydrodynamic variables. This point of view will be developed further in Section 5, within the framework of inhomogeneous world models. Here we only mention the following: In the general case, so long as the inequality $\upsilon \geq H$ holds, the forgetting of the initial distribution function is always essential for justifying a series of assumptions made in the development of hydrodynamic theories that involve finitely many degrees of freedom.

4. LINEARIZATION PROCEDURE

4.1. Preliminaries

In an exact description, the Einstein–Krook system would become a complicated set of integrodifferential equations for the evaluation of R and f. At the late stages of cosmic expansion, another device worth noting is that of using perturbation theory to obtain the linearized Einstein–Krook system. Let us see a little more explicitly how the formalism works for Robertson–Walker universe models. Given (3.11) and (3.14), we first define $\psi(t, z)$ and $\psi_E(t, z)$ by

$$\psi := F^{-1}(f - F) \tag{4.1a}$$

and

$$\psi_E := F^{-1}(\mathrm{F} - F) = \Omega \exp\left\{z^2\left[1 - \frac{2\Gamma}{1 + (1 + 2\kappa z^2)^{1/2}}\right]\right\} - 1 \tag{4.1b}$$

Clearly, these are natural measures of the deviation of f and $\mathbb{F}$ from a Maxwellian molecular density.

From the above characterization plus our notation introduced in the previous sections, equations (2.6) and (2.7) for the energy density e and the pressure p may be written as

$$\frac{e}{mc^2n} = 4\pi^{-1/2} \int_0^\infty z^2(1 + 2\kappa z^2)^{1/2} \exp(-z^2) [1 + \psi(t, z)] \, dz \qquad (4.2a)$$

$$\frac{p}{mc^2n} = \tfrac{8}{3} \pi^{-1/2}\kappa \int_0^\infty z^4(1 + 2\kappa z^2)^{-1/2} \exp(-z^2) [1 + \psi(t, z)] \, dz \qquad (4.2b)$$

In addition to (4.2), there is a corresponding equation for the number density n. Thus, using (3.14) and (4.1a), it follows at once that

$$n = 4\pi \int_0^\infty q^2 f \, dq = 4\pi^{-1/2} n \int_0^\infty z^2 \exp(-z^2) [1 + \psi(t, z)] \, dz \qquad (4.3)$$

With the scalar product

$$\langle \varphi_1, \varphi_2 \rangle := 4\pi^{-1/2} \int_0^\infty z^2 \exp(-z^2) \, \varphi_1(z)\varphi_2(z) \, dz \qquad (4.4)$$

relation (4.3) leads to the condition

$$\langle 1, \psi \rangle = 0 \qquad (4.5)$$

This condition is nicely consistent with the kinetic equation for ψ. Indeed, one can use (2.5), (2.6), (3.7), and (4.1) to obtain

$$\frac{\partial \psi}{\partial t} = \upsilon(\psi_E - \psi) \qquad (4.6)$$

and

$$\langle 1, \psi \rangle = \langle 1, \psi_E \rangle \qquad (4.7a)$$

$$\langle (1 + 2\kappa z^2)^{1/2}, \psi \rangle = \langle (1 + 2\kappa z^2)^{1/2}, \psi_E \rangle \qquad (4.7b)$$

in which case one concludes that $\langle 1, \psi \rangle$ is a constant. In other words, assuming only that $\langle 1, \psi \rangle$ vanishes initially, as is always possible, it is enough to guarantee the fulfillment of the constraint (4.5) for all times. Clearly, equations (4.7) may be viewed as being equivalent to equations (2.5). By substituting (4.1b) into (4.7) one then sees that Ω and Γ depend on time through $\kappa := k_B T/mc^2$ and ψ:

$$\Omega = \Omega(\kappa, \psi) \qquad (4.8a)$$

$$\Gamma = \Gamma(\kappa, \psi) \qquad (4.8b)$$

So far, we have done no more than introduce some useful definitions and transformations. However, consider a situation in which $\kappa^{1/4}$ and ψ are small. Under these circumstances, we may resort to perturbation theory. Here we do so by linearizing the expressions on the right-hand side of (4.2) and (4.8) with respect to κ and ψ:

$$(mc^2n)^{-1}e = 1 + \langle 1, \psi \rangle + \frac{3}{2}\kappa + \cdots \tag{4.9a}$$

$$(mc^2n)^{-1}p = \kappa + \cdots \tag{4.9b}$$

$$\Omega = 1 + \frac{15}{8}\kappa + \frac{5}{2}\langle 1, \psi \rangle - \langle z^2, \psi \rangle + \cdots \tag{4.10a}$$

$$\Gamma = 1 + \frac{5}{2}\kappa + \langle 1, \psi \rangle - \frac{2}{3}\langle z^2, \psi \rangle + \cdots \tag{4.10b}$$

Suppose now that $\langle 1, \psi \rangle$ and $\langle z^2, \psi \rangle$ are of the order κ. [Since $\langle 1, \psi \rangle = \text{const}$ and $\langle z^2, \psi \rangle$ is approximately independent of the time (see Section 4.2), we can also assume that $\langle 1, \psi \rangle = 0$ and $\langle z^2, \psi \rangle = 0$.] In the nonrelativistic range of temperatures, $q := (p^r p_r)^{1/2}$ is about $(2mk_BT)^{1/2}$ for the vast majority of the particles. From (3.6) and $\kappa^{1/4} << 1$ we then conclude that the typical value of z is very much smaller than $\kappa^{-1/s}$, where $s = 2, 4$. Thus $\kappa z^s << 1$ and the function ψ_E as given by (4.1b) simplifies to

$$\psi_E(t, z) = \frac{1}{2}\kappa(t)Q(z) + \langle 1, \psi(t) \rangle \left(\frac{5}{2} - z^2\right) - \langle z^2, \psi(t) \rangle \left(1 - \frac{2}{3}z^2\right) + \cdots \tag{4.11}$$

where

$$Q(z) := \frac{15}{4} - 5z^2 + z^4 \tag{4.12}$$

The manner in which these results form the lowest stage of an approximation procedure will become clear below.

4.2. The Asymptotic System of Equations

Applying (4.11) to (4.6), one obtain an equation of the form

$$\frac{\partial \psi}{\partial t} = \upsilon \left\{ \frac{1}{2}\kappa L[Q] - L[\psi] \right\} \tag{4.13}$$

where L denotes the linear operator whose action on φ is characterized by

$$L[\varphi] := \varphi - \langle 1, \varphi \rangle \left(\frac{5}{2} - z^2 \right) + \langle z^2, \varphi \rangle \left(1 - \frac{2}{3} z^2 \right) \tag{4.14}$$

For two arbitrary functions φ_1 and φ_2, we have

$$\langle \varphi_1, L[\varphi_2] \rangle = \langle L[\varphi_1], \varphi_2 \rangle \tag{4.15}$$

as a consequence of (4.4) and (4.14). Moreover, the independent, isotropic solutions of the homogeneous equation $L[\varphi] = 0$ are the classical collision invariants 1 and z^2. Finally, it is important to note that, because Q is orthogonal to 1 and z^2, the quantities $\langle 1, \varphi \rangle$ and $\langle z^2, \psi \rangle$ are constants; here, of course, ψ satisfies (4.13).

We have thus obtained a linearized propagation equation for ψ which exhibits the same basic features as the corresponding Boltzmann equation [see, e.g., Banach and Piekarski (1994a), equation (3.12)]. Even more, to convert all the previous calculations concerning ψ to the present context, one must only observe that here L is defined by (4.14), while in Boltzmann's theory L is the true linearized collision operator. Hence when both $\langle 1, \psi \rangle$ and $\langle z^2, \psi \rangle$ vanish initially, the kinetic-theory condition yields

$$|\psi(t, z)| \le \frac{1}{2} \kappa(t) |Q(z)| + [|\psi(t_0, z)| + \kappa(t_0)|Q(z)|] \frac{R(t_0)}{R(t)} \tag{4.16}$$

This result is very much analogous to the result (3.22a). Specifically, if the inequality $\upsilon \ge H$ holds for all times and if the function $R(t)$ increases with increasing t, then ψ evolves toward zero. Such is indeed the case, because equations (3.3) and (3.10) give $\kappa(t) \sim R^{-2}(t)$.

It is also straightforward to determine the form of an asymptotic expression for $\dot{\Gamma}(t)$. More explicitly, differentiating equation (4.10b) with respect to time, one sees immediately that

$$\dot{\Gamma}(t) = -5\kappa(t)H(t) \tag{4.17}$$

In obtaining (4.17) use was made of equations (3.3), (3.10), and (4.13). Here is the best place to mention the following: At the late stages of cosmic expansion, the results (4.10b) and (4.17) suggest that

$$\Gamma(t) \cong 1, \qquad |\dot{\Gamma}(t)| \cong 5\kappa(t)H(t) \tag{4.18}$$

and hence by (3.21a) and (3.21b) that

$$\mathbb{B} \cong 5, \qquad \mathbb{C} \cong 1 \tag{4.19}$$

Thus the theory of (4.13) is in many respects parallel to that of (3.7), which has been already considered.

On substituting (3.3) into (3.10), we find that the normalized temperature κ is related to R by $\kappa = b/mc^2R^2$. Since we are here considering the case when κ and ψ are small in some suitable sense, the energy density e and the pressure p will take the form

$$e = mc^2n\left(1 + \langle 1, \psi\rangle + \frac{3}{2}\kappa\right) \tag{4.20a}$$

$$p = mc^2n\kappa \tag{4.20b}$$

where n evolves at the same rate as R^{-3}, i.e., nR^3 = const. Because of (4.13), it is also possible to verify by direct substitution that (4.20) is a solution to equation (2.9). The result of differentiating the first of equations (2.8) and using (2.9) is equation (2.8b). Given (4.9) and (4.20), we now see that the linearized Einstein–Krook system, which governs the temporal evolution of R and ψ, consists of (2.8a) and (4.13). This system can easily be analyzed, especially in the case of a $k = 0$ Robertson–Walker geometry.

5. INDICATION OF THE DIRECTION OF FUTURE RESEARCH

The simplest cosmological models are those which are isotropic and spatially homogeneous, and it is natural to look at these first. However, if such models are a good approximation to the large-scale geometry of space-time in the region that we can observe, then the next step in the analysis is to provide a systematic framework for studying the time development of irregularities in the matter distribution. The full nonlinear Einstein–Boltzmann system is in principle the starting point for all work on these problems, but a hint of what to expect may also be obtained from consideration of Einstein's field equations and the relaxation-time approximation for a Boltzmann gas.

Clearly, the resulting system of equations is very complicated and has an extremely rich variety of solutions because of its nonlinear character. Nevertheless, in one limiting case it becomes fairly simple, namely, when the matter is only slightly perturbed away from the background Robertson–Walker model containing a pressure-free gas. Indeed, after replacing the system of nonlinear equations by its linearized approximation, we can prove the following result (Banach and Piekarski, 1994c): If the pressure vanishes in the background, then the treatment of gases by means of the linearized Einstein–Boltzmann system automatically produces a complete scheme of hydrodynamics, consisting of a closed set of partial differential equations for the evaluation of the mean velocity, the mass density, the temperature or the pressure, and the metric.

The time development of hydrodynamic variables would be completely independent of the wavelength of the perturbation if this condition were exactly fulfilled. Thus, in particular, Jeans' discussion of gravitational instability does not apply to the case considered here (vanishing pressure), for the temporal evolution is completely unaffected by particle horizons. Suppose, however, that the pressure is nonzero in the background. Then the following natural problem arises: Can one find exact solutions of the linearized Einstein–Boltzmann system which behave in an essentially hydrodynamic way and which tend to oscillate as a sound wave when the wavelength is less than the Jeans length? The conventional approaches focus exclusively on discussing Einstein's theory of gravitation in a small number of variables. Of course, since the Einstein–Boltzmann system involves infinitely many degrees of freedom, it remains to be seen how useful these simplified approaches will prove here. While perhaps some deeper theory would not imply that the resulting physical effects were large, it nevertheless seems an important question of principle.

Various generalizations of the present method will always show that a derivation of the questions of hydrodynamics is linked to the forgetting of the initial distribution function in the way assumed in Section 3.3. Thus the kinetic-theory condition is very essential for justifying a series of assumptions made in the development of phenomenological theories. To sum up, even though the inequality $\upsilon \geq H$ can hold only for rather special (but hopefully physically relevant) situations, the assumptions of this kind are absolutely necessary in order to explain the Jeans criterion via a standard hydrodynamic argument.

For the original Einstein–Boltzmann system, it is clear that much work remains to be done to develop these ideas into a fully effective tool. The situation would be radically different if strong analytical use could be made of the specific form of the relaxation-time collision term in (2.3). This has not yet been done, and the most significant application of the Einstein–Krook system would seem to be in problems which require a systematic derivation of the Jeans criterion for gravitational collapse.

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